Invariant tubular neighborhood theorem for affine varieties. *

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Abstract

The aim of this note is to prove the algebraic geometry analogue of the Invariant tubular neighborhood theorem which concerns the actions of compact Lie groups on smooth manifolds.

Introduction.

Let G be a compact Lie group acting on a smooth manifold M. Suppose $A \subset M$ is a smooth invariant closed manifold. Then it is well known ([4]) that A admits an open invariant tubular neighborhood in M. This means that there exist a (smooth) G-vector bundle E on A and an equivariant diffeomorphism $\phi\colon E \to M$ onto some open neighborhood of A in M such that the restriction of ϕ to the zero section of E is the inclusion of A in M. In this paper we prove in the case of linear actions of reductive groups on affine varieties the following analogue of the Invariant tubular neighborhood theorem.

Theorem: Let G be a reductive group acting linearly on an affine space A_k^n (ch k=0). Suppose $X \subset A_k^n$ is an affine smooth G-invariant subvariety. Then there exist (an algebraic) G-vector bundle E on X, a G-invariant open $U \subset E$

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which contains the zero section of E and $\phi\colon U \to A_k^n$ such that ϕ is etale, equivariant and its restriction to the zero section of E is the inclusion of X in A_k^n. The well known Luna Slice Theorem([7]) concerns the case when X is the closed orbit and states that ϕ can be chosen to be strongly etale.This does not seem to be true in the general case.

In the sequel all the varieties considered are over an algebraically closed field k with ch k=0. G will always denote a reductive group over k.

Preliminaries.

We shall need some elementary results about rational modules and modules of differentials in the equivariant context.

Let R be an affine k-algebra with a rational action of G and let M be an R-module such that G acts on M rationally.

Definition 1. *M is called an $R-G$ module if $g(rm) = g(r)g(m)$ for all $g \in G, r \in R$ and $m \in M$.*

For the proof of the following Proposition we refer to ([1]).

Proposition 1. *Let*

$$0 \to M^{'} \to M \to M^{''} \to 0 \tag{1}$$

be an exact sequence of $R-G$ modules with $M^{''}$ finitely generated. If (1) splits as a sequence of R-modules then it splits as a sequence of $R-G$ modules.

Let $\Omega_{R/k}$ denote the module of differentials. One can check easily that the formula $g(dr) = d(g(r))$ for $g \in G, r \in R$ defines an $R-G$ module structure on $\Omega_{R/k}$.

Suppose now that G acts linearly on $k[X_1, X_2, \dots X_n]$ and I is a G-invariant ideal of $A = k[X_1, X_2, \dots X_n]$. Then the induced action of G on $R = A/I$ is rational and R is obviously an affine k-algebra.

Proposition 2. *Let R be smooth over k.Then the first fundamental exact sequence ([5])*

$$0 \to I/I^2 \to \Omega_{A/k} \otimes A/I \to \Omega_{R/k} \to 0 \tag{2}$$

splits as a sequence of $R-G$ modules.

Proof. Note that all the maps in (2) are G-invariant. Moreover (2) is a splitting sequence of R-modules since R is smooth. We can now apply Proposition 1. $\square$

In what follows I is an ideal of a ring A (commutative and noetherian) and we put $\widehat{A} = \varprojlim A/I^n$

Lemma . *([3]) Let $I \subset A$ be an ideal and let $\mathfrak{m} \supset I$ be a maximal ideal of A. Put $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{A}$. Then the map $A_{\mathfrak{m}} \to \hat{A}_{\widehat{\mathfrak{m}}}$ induces the isomorphism of the completion of $A_{\mathfrak{m}}$ with respect to the $\mathfrak{m}A_{\mathfrak{m}}$-adic topology with the completion of $\hat{A}_{\widehat{\mathfrak{m}}}$ with respect to the $\widehat{\mathfrak{m}}\hat{A}_{\widehat{\mathfrak{m}}}$-topology.*

We immediately obtain the following

Proposition 3. *Let G act rationally on affine k-algebras A and B with G-invariant ideals I and J respectively. Suppose $d\colon A \to B$ is a G-invariant homomorphism which for all n induces an isomorphism $A/I^n \simeq B/J^n$.Then the set of all the points of $Spec\, B$ at which the induced map $Spec\, B \to Spec\, A$ is etale is an open, G-invariant subset containing all the closed points $[\mathfrak{m}]$ such that $\mathfrak{m} \supset J$*

Proof. It suffices to note that the morphism $Spec\, B \to Spec\, A$ is etale at x=$[\mathfrak{m}]$ if and only if f induces the isomorphism of the completions of the corresponding local rings ([8]). Moreover the "etale" property is local and G-invariant for formal reasons since d is G-invariant. $\square$

Main Theorem.

Theorem . *Let G act linearly on an affine space A^n_k. Suppose $X \subset A^n_k$ is a closed, affine and smooth G-invariant subvariety. Then there exist (an algebraic) G-vector bundle E on X, a G-invariant open $U \subset E$ which contains the zero section of E and $\phi\colon U \to A^n_k$ such that ϕ is etale, G-invariant and its restriction to the zero section of E is the inclusion of X in A^n_k.*

Proof. We put $A = k[X_1, X_2, \dots X_n]$ and $R = A/I$ where I is a G- invariant ideal of A which corresponds to X. Then by Proposition 2 there exists a $R{-}G$ homomorphism $\gamma\colon \Omega_{A/k} \otimes A/I \to I/I^2$ which splits $I/I^2 \to \Omega_{A/k} \otimes\ A/I$. For $f \in A$ we put $d_1(f) = \gamma(\overline{df})$ where $\overline{df}$ denotes the image of df in $\Omega_{A/k} \otimes A/I$. Thus $d_1\colon A \to I/I^2$ is G-invariant and its restriction to I/I^2 coincides

with the natural homomorphism $I \to I/I^2$. We denote by d_0 the natural homomorphism $A \to A/I$.

The association $X_i \to d_0(X_i) + d_1(X_i)$ for $i = 1, 2, \dots, n$ defines $d\colon A \to S(I/I^2)$ where $S(I/I^2)$ denotes the symmetric algebra of the R-module I/I^2 with the obviously induced action of G. It turns out that $d\colon A \to S(I/I^2)$ is a G-invariant k-algebra homomorphism.

Let $d_i\colon A \to S^i(I/I^2) = I^i/I^{i+1}$ denote the i-th component of d. It is easy to check that d_0 and d_1 coincide with the previously defined d_0 and d_1. Moreover $d_i(fg) = \sum_{j+k=i} d_j(f)d_k(g)$ for $f, g \in A$ since d is an algebra homomorphism.

We claim that d_i restricted to I^i is the natural homomorphism $I^i \to I^i/I^{i+1}$ for $i \geq 0$. This is true for $i = 0$ and $i = 1$. Let $f \in I^{i-1}$ and $g \in I$. By induction

$$\begin{aligned} d_i(fg) = d_{i-1}(f)d_1(g) &= (\text{ image of } f \text{ in } I^{i-1}/I^i)(\text{ image of g } \text{ in } I/I^2) \\ &= \text{ image of } fg \text{ in } I^i/I^{i+1}. \end{aligned}$$

This proves the claim since I^i is additively generated by the elements of the form fg with $f \in I^{i-1}$ and $g \in I$.

We put $J = \bigoplus_{i \geq 1} S^i(I/I^2)$ which is a G-invariant ideal of $S(I/I^2)$. Obviously for all n $\;d(I^n) \subset J^n$. It follows that the induced homomorphism $A/I^n \to S(I/I^2)/J^n$ is injective for all n.

To prove its surjectivity it suffices to show that the homogenous elements of $\bigoplus_{i \leq n-1} I^i/I^{i+1}$ differ from the image of d by the elements of $\bigoplus_{i \geq n} I^i/I^{i+1}$. Let $x \in \bigoplus_{i \leq n-1} I^i/I^{i+1}$ with $\deg x = k \quad (k \leq n-1)$. Then there exists $f \in I^k$ such that $d_k(f) = x$. We shall define inductively a sequence of elements $\{f_i\}_{k \leq i \leq n-1}$ with $f_i \in I^i$. Put $f_k = f$. Suppose the sequence $f_k, f_{k+1}, \dots f_i$ with $f_j \in I^j \quad k \leq j \leq i$ has been defined. Let f_{i+1} be an element of I^{i+1} such that $d_{i+1}(f_{i+1}) = -d_{i+1}(f_k + f_{k+1} + \cdots + f_i)$. Then $x - d(\sum_{i=k}^{n-1} f_i) \in \bigoplus_{i \geq n} I^i/I^{i+1}$. So the induced map $A/I^n \to S(I/I^2)/J^n$ is an isomorphism for all n.

We denote with U the set of all the closed points of $Spec\, S(I/I^2)$ at which the morphism induced by d $Spec\, S(I/I^2) \to Spec\, A$ is etale. Then by Proposition 3 $\;U$ is a G-invariant open subset of E containing its zero section where E denotes the normal bundle of X in A_k^n i.e the set of the closed points of $Spec\, S(I/I^2)$. Moreover $Spec\, S(I/I^2) \to Spec\, A$ maps U into A_k^n which is the set of the closed points of $Spec\, A$. The restriction of the obtained $\phi\colon U \to A_k^n$ to the zero section of E is an identity on X since $d\colon A \to S(I/I^2)$

induces the identity homomorphism $R = A/I \to S(I/I^2)/J = R$. Thus $\phi\colon U \to A_k^n$ has all the required properties. $\square$

Remark 1. *The existence of* $d\colon A \to S(I/I^2)$ *which induces the isomorphism* $A/I^n \simeq S(I/I^2)/J^n$ *for all* n *has already been proved in the "absolute" case in [2]. The proof actually shows that a* k*-algebra homomorphism* $d = (d_i)\colon A \to S(I/I^2)$ *induces the isomorphism* $A/I^n \simeq S(I/I^2)/J^n$ *for all* n *if* $d_0\colon A \to A/I$ *is the natural homomorphism and the restriction of* $d_1\colon A \to I/I^2$ *to* I *is the natural homomorphism* $I \to I/I^2$.

Remark 2. *One easily obtains the following version of the Main Theorem in case the action of* G *is not necessarily linear.*
Let G *be a reductive group acting (rationally) on an affine variety* Y *and let* $X \subset Y$ *be a closed, affine and smooth* G*-invariant subvariety. Then there exist (an algebraic)* G*-vector bundle* E *on* X*, a* G*-invariant locally closed* $U \subset E$ *which contains the zero section of* E *and* $\phi\colon U \to Y$ *such that* ϕ *is etale,* G*-invariant and its restriction to the zero section of* E *is the inclusion of* X *in* Y.
Proof: Y *is isomorhic to a* G*- invariant closed affine subvariety of* A_k^n *with a linear action of* G *([6]). Then by the Main Theorem there exist (an algebraic)* G*-vector bundle* E *on* X*, a* G*-invariant open* $U_1 \subset E$ *which contains the zero section of* E *and* $\phi\colon U_1 \to A_k^n$ *such that* ϕ *is etale,* G*-invariant and its restriction to the zero section of* E *is the inclusion of* X *in* A_k^n*. Put* $U = \phi^{-1}(Y)$*. Then* $U \subset E$ *and* $\phi|U\colon U \to Y$ *have all the required properties.*

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